

Mean field theory for skewed height profiles in KPZ growth processes

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Abstract

We propose a mean field theory for interfaces growing according to the Kardar-Parisi-Zhang (KPZ) equation in 1+1 dimensions. The mean field equations are formulated in terms of densities at different heights, taking surface tension and the influence of the nonlinear term in the KPZ equation into account. Although spatial correlations are neglected, the mean field equations still reflect the spatial dimensionality of the system. In the special case of Edwards-Wilkinson growth, our mean field theory correctly reproduces all features. In the presence of a nonlinear term one observes a crossover to a KPZ-like behavior with the correct dynamical exponent $z = 3/2$. In particular we compute the skewed interface profile during roughening, and we study the influence of a co-moving reflecting wall, which has been discussed recently in the context of nonequilibrium wetting and synchronization transitions. Also here the mean field approximation reproduces all qualitative features of the full KPZ equation, although with different values of the surface exponents.

1 Introduction

Since many years the physical properties of deposition-evaporation processes on a planar surface have been studied theoretically by analyzing appropriate

stochastic growth models that capture the essential features of the experimental realm [1]. In most of these models the configuration of the growing surface is described by a height variable $h(\vec{x}, t)$ that yields the height of the interface between deposited layer and gas phase above point \vec{x} of the substrate at time t . Starting with a certain initial configuration, the interface then evolves according to certain stochastic rules.

Depending on the specific dynamic rules for deposition and evaporation and their symmetries, the temporal evolution of the interface may be described on a coarse-grained scale by a stochastic differential equation, one of the simplest and more general one being the celebrated Kardar-Parisi-Zhang (KPZ) equation [2]

$$\frac{\partial}{\partial t} h(\vec{x}, t) = v_0 + D \nabla^2 h(\vec{x}, t) + \frac{\lambda}{2} [\nabla h(\vec{x}, t)]^2 + \xi(\vec{x}, t). \quad (1)$$

Here v_0 is the average growth velocity which can be set to zero in a co-moving frame, the Laplacian accounts for surface tension of the interface, and $\xi(\vec{x}, t)$ is an uncorrelated white Gaussian noise generated by the stochastic nature of deposition and evaporation. Moreover, the nonlinear term $(\nabla h)^2$ it's the simplest one which breaks the invariance under reflections $h \rightarrow -h$.

As many models for interface growth, the KPZ equation exhibits dynamic scaling, i.e., starting with a flat configuration the interface width $w(t) = \sqrt{\langle h^2 \rangle - \langle h \rangle^2}$ (where $\langle \cdot \rangle$ denotes average over space and ensemble realizations) first increases as a power law $w(t) \sim t^\gamma$ until it saturates in a finite system of linear size L at a stationary value $w_{\text{stat}} \sim L^\alpha$. The crossover from a roughening to a stationary state is described by the well-known Family-Vicsek scaling form [3]

$$w(t) \sim t^\gamma g(t/L^z), \quad (2)$$

where g is a universal scaling function and $z = \alpha/\gamma$ is the dynamical exponent.

The width is actually related to the second moment of the height distribution profile $P_L(h, t)$, which is defined as the normalized probability to find the interface at a randomly chosen lattice site at height h . Clearly, the height distribution contains much more information about the interface morphology than the width alone. As for the width, dynamic scaling implies a scaling form for the height distribution which in a co-moving frame may

be written as¹

$$P_L(h, t) = t^{-\gamma} f(h/t^\gamma, t/L^z). \quad (3)$$

Obviously, both the critical exponents α and γ and the shape of height profile during roughening or after saturation reflect the symmetries of the growth process under consideration. The simple case of invariance under the reflection $h \rightarrow -h$ can be studied by imposing $\lambda = 0$ in Eq. (1). In this case the linear Edwards-Wilkinson (EW) equation [4] is recovered, and the critical growth exponents in 1+1 dimensions take the values $\alpha = 1/2$ and $\gamma = 1/4$. Moreover, the height profile of 1+1-dimensional EW processes is known to be a simple Gaussian distribution both in the dynamically roughening phase as well as in the stationary state.

In more realistic growth models, where nearest neighbour interactions play a role in the dynamics of the growing interface, reflection symmetry is broken and the nonlinear KPZ term has to be taken into account. In 1+1 dimensions such a term is known to be *relevant* in the renormalization group sense. Therefore, even when the reflection symmetry is weakly violated (i.e., if λ is small), the scaling behavior of an infinite system will eventually cross over from EW to KPZ scaling, the latter being characterized by the exponents $\alpha = 1/2$, $\gamma = 1/3$, and $z = \alpha/\gamma = 3/2$.

With a non-symmetric term being present there is no longer any reason for the height distribution $P_L(h, t)$ to be symmetric with respect to h . Although in 1+1 dimensions a KPZ interface of a *finite* system after saturation still happens to be symmetric and Gaussian (see e.g. [5]), the profile of a *roughening* KPZ interface before saturation is indeed skewed [6], reflecting the asymmetry of the nonlinear term. In what follows we therefore restrict ourselves to the roughening process before saturation, i.e., $t \ll L^z$, regarding a virtually infinite system at finite times. Formally speaking, this can be achieved by taking the thermodynamic limit $L \rightarrow \infty$ *before* the time-asymptotic limit $t \rightarrow \infty$ is carried out. In particular, we are interested in the scaling function $f(z) := f(z, 0)$, which renders the rescaled shape of the skewed profile after sufficiently long time (see left panel of Fig. 1).

The function $f(z)$ is known to be universal, i.e., the asymptotic shape of the skewed profile is fully determined by the underlying KPZ field theory and does not depend on the microscopic details of the model. It has been suggested that the finite-time height distribution, especially the form of its

¹The scaling functions g and f are related by $g^2(u) = \int_0^\infty f(u, v)v^2 dv - [\int_0^\infty f(v, u)v dv]^2$.

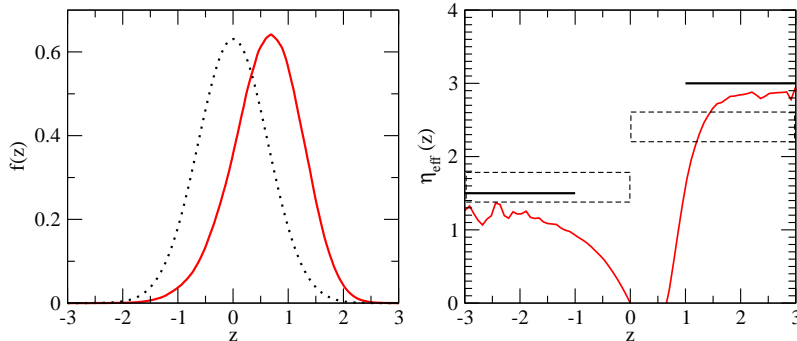


Figure 1: Skewed interface profile. Left panel: Numerically determined profile $f(z) := f(z, 0)$ of a roughening KPZ interface in 1+1 dimensions for $\lambda < 0$ (solid line) compared to a Gaussian distribution of the same width (dashed line). Right panel: Effective exponents $\eta_{eff}(z)$ (see text) in comparison with the numerical estimates $\eta_+ = 1.6(2)$ and $\eta_- = 2.4(2)$ reported in Ref. [10] (representing the error bars as dashed boxes) and the theoretical predictions for the PNG model [8] (marked by full horizontal lines).

tails, is approximately given by a stretched exponential

$$P_L(h, t) \propto \exp[-\mu(|h - \langle h \rangle|/t^\gamma)^{\eta_\pm}] \quad t \ll L^z, \quad (4)$$

meaning that $f(z) \sim \exp(-\mu|z|^{\eta_\pm})$. Here μ is a metric factor while the exponents η_\pm refer to the two different tails of the distribution with $\pm\lambda(h - \langle h \rangle) > 0$. Because of the skewness both exponents are expected to be different. An argument based on a replica scaling analysis of directed polymers [7], whose free energy fluctuations corresponds to the height fluctuations of a KPZ interface, suggests the value $\eta_+ = 3/2$.

As a breakthrough, Prähofer and Spohn have shown recently [8] that the finite-time rescaled height profile of the polynuclear growth model (PNG) [9], a model which is believed to belong to the KPZ universality class, equals the Gaussian orthogonal ensemble (GOE) Tracy-Widom distribution. This immediately leads to

$$\eta_+ = 3/2, \quad \eta_- = 3. \quad (5)$$

Numerical simulations reported in literature concerning both directed polymers and KPZ lattice models [10] give $\eta_+ = 1.6(2)$ and $\eta_- = 2.4(2)$, the latter value being not in agreement with theoretical predictions. However, since these results were obtained more than a decade ago the numerical precision was limited. Performing similar simulations using the so-called

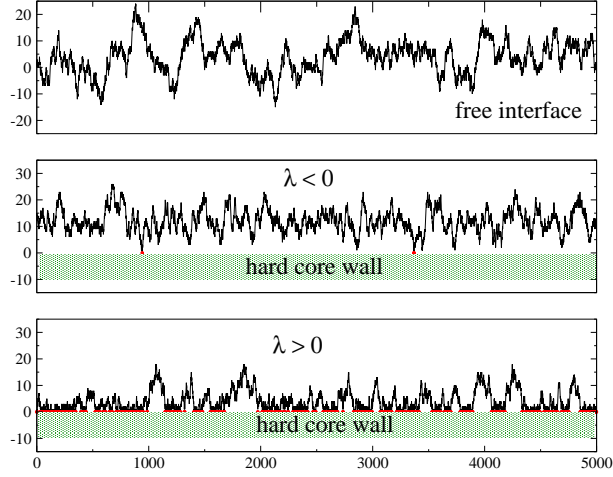


Figure 2: Snapshots of a roughening interface in a reference frame where the asymptotic velocity is zero. The figure shows a free interface with $\lambda < 0$ (upper panel) compared to interfaces confined by a co-moving lower wall in the two cases $\lambda < 0$ (middle panel) and $\lambda > 0$ (lower panel). Simulations have been performed using the single step model (see Sec. 2) and snapshots have been taken after 2048 time steps.

single step model [11] (see Sec. 2) we measured the effective exponent $\eta_{\text{eff}}(z) = \frac{z}{\ln f(z)} \frac{d}{dz} \ln f(z)$ which according to Eq. (4) should converge to η_{\pm} as $|z| \rightarrow \infty$. As it can be seen in the right panel of Fig. 1, our numerical results are in agreement with theoretical predictions for the PNG model while being incompatible with the previous estimate for η_- of Ref. [10]. Clearly, the center of the scaling function (i.e. small values of z) is described only approximately by Eq. (4).

Looking at a snapshot of a roughening interface in 1+1 dimensions, it is almost impossible to recognize the influence of the KPZ nonlinearity by naked eye. Its influence, however, is much more pronounced in the presence of a hard-core wall. The wall is fixed in a frame where the asymptotic velocity of the interface vanishes and interacts with the interface solely by preventing excursions to negative heights. As can be seen in Fig. 2, in presence of such a wall one can easily appreciate the dramatic difference emerging when the sign of λ is changed². Surprisingly, for $\lambda < 0$ the interface touches the wall only occasionally, while a high density of contact points is

²Alternatively one may compare a lower and an upper wall while keeping the sign of λ fixed.

observed for $\lambda > 0$.

The properties of a KPZ interface close to a reflecting wall has been studied recently in the context of nonequilibrium wetting [12], where the interface describes a wetting layer on a planar substrate. Upon varying the average growth velocity v_0 the interface undergoes a *depinning* transition between a pinned phase, in which portions of the interface remains attached to the wall, to a depinned phase, where the interfaces detaches entirely and starts moving upwards. At the critical point, where the asymptotic interface velocity is zero, a second order phase transition takes place and various scaling laws can be singled out. In addition, the case $\lambda < 0$ describes the critical properties of most synchronization transitions in spatially extended chaotic systems [13, 14].

Previous numerical simulations suggested that the temporal decay of the density $\rho_0(t)$ of contact points, where the interface touches the wall, obeys the power law [15, 16]

$$\rho_0(t) \sim t^{-\theta}, \quad \theta \approx \begin{cases} 1.1(1) & \text{if } \lambda < 0 \\ 3/4 & \text{if } \lambda = 0 \\ 0.22(2) & \text{if } \lambda > 0 \end{cases}, \quad (6)$$

where the exponent $3/4$ can be obtained from a transfer matrix calculation [12]. Moreover, a hyperscaling relation observed in simulations starting with a single pinned site [17] suggests the rational value $\theta = 7/6$. Obviously, the different values of the exponents reflect the asymmetry of the nonlinear term with respect to reflections $h \rightarrow -h$. Moreover, the pronounced numerical variation of θ by a factor of 5 explains why the snapshots in Fig. 2 are so strikingly different.

Interestingly, the profile of a roughening KPZ interface next to a wall cannot be described in terms of an appropriate generalized GOE Tracy-Widom distribution because of emerging nonlinear term. More generally, the critical behavior of such a bounded growth process in 1+1 dimensions is not easily accessible by analytical means. For example, renormalization group techniques fail either due to the presence of a strong-coupling fixed point inaccessible by perturbative approaches in the case $\lambda < 0$ [18] or due to essential singularities arising for $\lambda > 0$. Therefore, the primary aim of the present paper is to discuss this case within a suitable mean field approximation. The mean field theory to be constructed should incorporate the asymmetry caused by the nonlinear term and should render a skewed height distribution with similar properties as in the full model. Although the mean

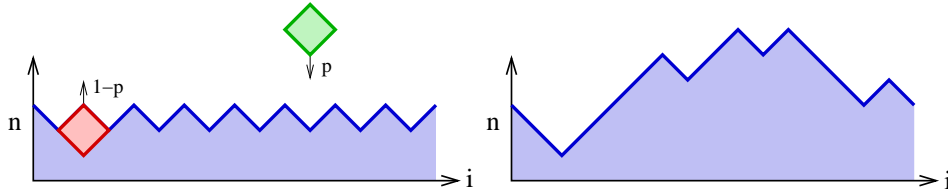


Figure 3: Single step model in 1+1 dimension. Left panel: The simulation usually starts with a flat interface, realized as a horizontal sawtooth pattern. On selecting a site with a local minimum a diamond (rhombus) is deposited with probability p , flipping up the interface by two units. Similarly, if the selected site happens to be a local maximum, a diamond may evaporate with probability $1 - p$, flipping the interface downward by two units. Right panel: For $p > \frac{1}{2}$ the interface roughens and propagates upwards.

field theory ignores space, it should not resemble a naive infinite-dimensional limit (where a KPZ interface is always smooth), instead it should reflect to some extent the dimensionality of space in the thermodynamic limit $L \rightarrow \infty$. Moreover, the desired theory should be as simple as possible and exactly solvable. In the following sections we propose and solve a mean field theory which meets these requirements.

2 Mean field equations

The mean field equations proposed here are inspired by a particular model, the so-called single step model (SSM) [11], which is probably the simplest and most compelling lattice model for KPZ-type interface growth.

In the single step model the growing interface is represented by a set of integer heights $n_i \in \mathbb{N}$ residing at the sites $i = 1 \dots L$ of a one-dimensional lattice of length L with periodic boundary conditions, obeying the restriction

$$n_{i+1} - n_i = \pm 1. \quad (7)$$

The interface evolves in time according random sequential updates as follows: At each sub-time step $dt = 1/L$ a site i is chosen at random. If the interface has a local minimum at site i (i.e., $n_i < n_{i\pm 1}$) the height n_i is increased by 2 with probability $p \in [0, 1]$. This update can be pictured as depositing a diamond (see Fig. 3), transforming a local minimum into a local maximum. Similarly, if the selected site happens to be a local maximum ($n_i > n_{i\pm 1}$) the height n_i is decreased by two units with probability $1 - p$.

For $p = 1/2$ the propagation velocity of the interface is zero and the evolution rules satisfy detailed balance, as described by the EW equation. For $p \neq 1/2$, however, the propagation velocity is nonzero, depending on the roughness and the average slope of the interface. In this case the SSM exhibits KPZ growth with λ being equal to $\frac{1}{2} - p$.

By identifying upward segments $n_{i+1} - n_i = 1$ with particles and downward segments $n_{i+1} - n_i = -1$ with vacancies, the single step model can be mapped exactly onto a partially asymmetric exclusion process (ASEP) [19] of diffusing particles with density $1/2$. Since the ASEP is known to evolve towards an uncorrelated product state with a current $j = p/2 - 1/4$, it is immediately clear that the propagation velocity of the interface tends to

$$v_\infty = \lim_{t \rightarrow \infty} v(t) = p - \frac{1}{2} \quad (8)$$

as $t \rightarrow \infty$.³ Moreover, the mapping to the ASEP allows one to solve the model via Bethe ansatz [20], making it possible to derive the KPZ dynamical exponent $z = 3/2$ and various other quantities exactly. Other rigorous results concerning shape fluctuations in the ASEP can also be found in Ref. [21].

In order to formulate a mean field theory for the single step model, let $N_u(n, t)$ and $N_d(n, t)$ be the probabilities of finding an upward or downward segment with their lower edge rooted at height level n . Let us first consider a deposition process, in which a local minimum at level n is flipped into a local maximum at level $n+2$. Having selected a random site, the probability to find such a local minimum at a given height can be approximated as follows. Clearly, the probability of finding a downward segment on the left side terminating at height n is $N_d(n, t)/L$, where L is the system size. With this probability, knowing that the height of the selected site is n , the adjacent segment to the right can only go up or down so that the *conditional* probability to find an upward segment is given by $N_u(n, t)/(N_u(n, t) + N_d(n-1, t))$. Ignoring possible correlations the total probability of finding a local minimum at height n is the product of these two expressions. The deposition process, taking place with probability p , therefore leads to a loss of probability at level n

$$\begin{aligned} N_u(n, t) \rightarrow N_u(n, t + dt) &= N_u(n, t) - \frac{p}{L} \frac{N_d(n, t)N_u(n, t)}{(N_u(n, t) + N_d(n-1, t))} \\ N_d(n, t) \rightarrow N_d(n, t + dt) &= N_d(n, t) - \frac{p}{L} \frac{N_d(n, t)N_u(n, t)}{(N_u(n, t) + N_d(n-1, t))} \end{aligned} \quad (9)$$

³Initially the velocity is higher, and the excess velocity $|v(t) - v_\infty|$ decays as $t^{-1/3}$.

and a corresponding gain at level $n + 1$. Similar expressions can be derived for the evaporation process. Obviously, this approximation accounts for the restriction (7) and the one-dimensional structure of the model but disregards possible nearest-neighbor correlations.

The structure of Eq. (9) suggests that the probabilities $N_u(n, t)$ and $N_d(n, t)$ evolve exactly in the same way. In fact, it is easy to see that in a system with periodic boundary conditions the numbers of upward and downward segments are exactly equal. Assuming the same to hold in an infinite system we have

$$N_u(n, t) = N_d(n, t) \quad (10)$$

for every n and t . Thus, introducing a combined probability density

$$P(n, t) = \frac{N_d(n, t) + N_u(n, t)}{2L} \quad (11)$$

the loss at level n due to a deposition event in Eq. (9) can be recast as

$$P(n, t) \rightarrow P(n, t + dt) = P(n, t) - p \frac{P_n(t)^2}{P_n(t) + P_{n-1}(t)}. \quad (12)$$

Collecting all loss and gain contributions due to deposition and evaporation one arrives at the following set of mean field equations

$$\begin{aligned} \frac{\partial}{\partial t} P_n(t) = & p \left[\frac{P_{n-1}(t)^2}{P_{n-1}(t) + P_{n-2}(t)} - \frac{P_n(t)^2}{P_n(t) + P_{n-1}(t)} \right] \\ & + (1 - p) \left[\frac{P_{n+1}(t)^2}{P_{n+1}(t) + P_{n+2}(t)} - \frac{P_n(t)^2}{P_n(t) + P_{n+1}(t)} \right] \end{aligned} \quad (13)$$

which serve as a starting point for all further calculations throughout this paper. Notice that the form of the denominators appearing on the r.h.s of Eq. (13) is a consequence of the restriction (7), and that some care has to be taken when one of them vanishes. Since the numerators are quadratic we assume that each of these terms is zero *whenever* their denominator vanishes.

Introducing a probability current flowing between neighboring levels

$$J_{n,n+1}(t) = p \frac{P_n(t)^2}{P_n(t) + P_{n-1}(t)} - (1 - p) \frac{P_{n+1}(t)^2}{P_{n+1}(t) + P_{n+2}(t)} \quad (14)$$

these equations can be also written as

$$\frac{\partial}{\partial t} P_n(t) = J_{n-1,n}(t) - J_{n,n+1}(t). \quad (15)$$

Obviously, they conserve probability $\sum_{k=-\infty}^{+\infty} P_k(t)$ so that the integrated probability distribution

$$Q_n(t) := \sum_{k=-\infty}^n P_k(t) \quad (16)$$

satisfies the simple evolution equation

$$\frac{\partial}{\partial t} Q_n(t) = -J_{n,n+1}(t). \quad (17)$$

By construction these mean field equations reflect both the one-dimensional structure as well as the restriction but they ignore spatial correlations between the segments. The full model does exhibit such correlations, but it evolves towards a trivial state without correlations (corresponding to a simple product state in the ASEP). Although this trivial state is never reached in an infinite system, it may explain why the mean field equations proposed here reproduce so many of the observed phenomena faithfully, some of them even exactly, as will be shown in the following sections.

3 Exact solution of the mean field equations

Let us first consider the case of a free interface, where the height index n runs over all integers from $-\infty$ to $+\infty$.

As the KPZ equation is invariant under appropriate rescaling of space, time and height [5], we can carry out the continuum limit by introducing a new height variable $h = n\Delta$, where Δ is the new height unit of the rescaled system. In order to investigate the asymptotic properties of the roughening processes let us assume that $P_n(t)$ varies only slowly with n and expand the r.h.s. of Eq. (13) as a Taylor series around h . Keeping contributions up to fourth order in Δ we obtain the partial differential equation

$$\begin{aligned} \frac{\partial}{\partial t} P(h, t) = & \frac{\Delta(1-2p)}{2} P'(h, t) + \\ & \frac{\Delta^3(2p-1)}{24} \left[\frac{3P'(h, t)^3}{P(h, t)^2} - \frac{6P'(h, t)P''(h, t)}{P(h, t)} + 4P'''(h, t) \right] + \\ & \frac{\Delta^4}{8} \left[\frac{2P'(h, t)^4}{P(h, t)^3} - \frac{5P'(h, t)^2P''(h, t)}{P(h, t)^2} + \right. \\ & \left. \frac{2P''(h, t)^2}{P(h, t)} + \frac{2P'(h, t)P'''(h, t)}{P(h, t)} - P''''(h, t) \right], \end{aligned} \quad (18)$$

where the prime stands for a partial derivative with respect to h . Obviously the leading term of order Δ on the r.h.s. generates a uniform propagation of the probability distribution, hence, the average height $\langle h(t) \rangle$ of the interface will asymptotically grow with the linear velocity

$$v = \Delta \left(p - \frac{1}{2} \right) \quad (19)$$

plus some sublinear correction terms. Assuming ordinary Family-Vicsek scaling [3], it is therefore near at hand to test the validity of the scaling form

$$P(h, t) = t^{-\gamma} f\left(\frac{h - vt}{t^\gamma}\right) \quad (20)$$

which – by definition – conserves the integrated probability $\int_{-\infty}^{+\infty} dh P(h, t)$. Notice that the normalization of the height probability distribution implies the scaling function $f(z)$ to be normalized as well. In what follows we solve Eq. (18) both in the equilibrium case $p = 1/2$ and the non-equilibrium case $p \neq 1/2$ confirming that our results do not depend on Δ . In particular, we will show that higher order terms appearing in the expansion of Eq. (13) turns to be irrelevant, vanishing in the asymptotic limit $t \rightarrow \infty$. The correct asymptotic behavior of Eq. (13) will be therefore recovered by setting $\Delta = 1$.

3.1 Equilibrium roughening of a free interface

We start analyzing the special case $p = 1/2$, where the dynamic processes of the full model are known to exhibit detailed balance. In this case the velocity v is zero and the first-order and third-order contributions on the r.h.s. of Eq. (18) vanish. Inserting the Ansatz (20) into Eq. (18) we find that, up to fourth order, the partial differential equation reduces to a non-trivial ordinary differential equation for the scaling function if and only if $\gamma_{\text{eq}} = 1/4$ (the subscript denoting the equilibrium case). This is exactly the value predicted by the EW theory for equilibrium roughening. Moreover one easily notices that by fixing $\gamma_{\text{eq}} = 1/4$ higher order terms $O(\Delta^5)$ occurring in the Taylor expansion of Eq. (13) are irrelevant in the asymptotic limit $t \rightarrow \infty$. The differential equation therefore reads

$$\begin{aligned} \frac{1}{f_{\text{eq}}(z)^3} \left[f_{\text{eq}}(z)^4 - \frac{5}{2} \Delta^4 f_{\text{eq}}(z) f'_{\text{eq}}(z)^2 f''_{\text{eq}}(z) + \right. \\ \left. \Delta^4 f_{\text{eq}}(z)^2 \left(f''_{\text{eq}}(z)^2 + f'_{\text{eq}}(z) f'''_{\text{eq}}(z) \right) + \right. \\ \left. f_{\text{eq}}(z)^3 \left(z f'_{\text{eq}}(z) - \frac{\Delta^4}{2} f'''_{\text{eq}}(z) \right) + \Delta^4 f'_{\text{eq}}(z)^4 \right] = 0, \end{aligned} \quad (21)$$

where $z = h t^{-\gamma_{\text{eq}}}$ denotes the scaling variable. Integrating both sides we obtain

$$z f_{\text{eq}}(z) - \frac{\Delta^4}{2} \left[\frac{f'_{\text{eq}}(z)^3}{f_{\text{eq}}(z)^2} - 2 \frac{f'_{\text{eq}}(z) f''_{\text{eq}}(z)}{f_{\text{eq}}(z)} + f'''_{\text{eq}}(z) \right] = 0. \quad (22)$$

This equation admits the two simple solutions

$$f_{\text{eq}}^{\text{free}}(z) = \frac{1}{\Delta 2^{1/4} \sqrt{\pi}} \exp \left(-\frac{z^2}{\Delta^2 \sqrt{2}} \right) \quad (23)$$

and

$$f_{\text{eq}}^{\text{bound}}(z) = \frac{2^{1/4}}{\Delta^3 \sqrt{\pi}} z^2 \exp \left(-\frac{z^2}{\Delta^2 \sqrt{2}} \right) \quad (24)$$

which have been normalized over the real line. The first solution $f_{\text{eq}}^{\text{free}}$ is a simple Gaussian and represents the physical solution for a free interface starting with a flat initial condition $h(x, t) = 0$. The second solution $f_{\text{eq}}^{\text{bound}}$ is characterized by two different maxima over the real line and is therefore dismissed as unphysical in the free case. However, as we will see in Sect. 4, this solution becomes physically meaningful in the presence of a hard-core wall at zero height.

To summarize we note that the mean field equation for $p = 1/2$ does indeed capture the features of one-dimensional EW roughening in the thermodynamic limit $L \rightarrow \infty$.

3.2 Nonequilibrium roughening of a free interface

We now turn to the nonequilibrium case $p \neq 1/2$. Inserting again the scaling form (20) and the expression for the velocity (19) into the partial differential equation (18), we find that, up to *third* order, the partial differential equation reduces to a non-trivial ordinary differential equation for the scaling function (i.e., without explicit occurrence of t) if and only if $\gamma = 1/3$. The differential equation then reads

$$\frac{1}{f(z)^2} \left[8f(z)^3 + 3kf'(z)^3 - 6kf(z)f'(z)f''(z) + 4f(z)^2(2zf'(z) + kf'''(z)) \right] = 0, \quad (25)$$

where $z = (h - vt) t^{-\gamma}$ denotes the scaling variable in the comoving reference frame and

$$k = (2p - 1)\Delta^3 \neq 0. \quad (26)$$

As in the equilibrium case the postulate of Family-Vicsek scaling applied to the mean field equation already determines the roughening exponent. Remarkably the value $\gamma = 1/3$ coincides exactly with the known value of a KPZ process in 1+1 dimensions.

As can be verified easily, upon fixing $\gamma = 1/3$, the fourth order terms (and all higher order terms) of the Taylor expansion turn out to be irrelevant in the asymptotic limit $t \rightarrow \infty$ and thus generate only short time corrections to the scaling function. It is also worth commenting that asymptotic EW scaling can only be seen in the symmetric case $p = 1/2$. For any small deviation from this value the third-order terms do not vanish, leading eventually to a crossover to KPZ scaling in the limit $t \rightarrow \infty$. Therefore, the mean field equations nicely reproduce the character of the KPZ nonlinearity as a relevant perturbation.

Assuming that $f(z) \neq 0$ and integrating both sides of Eq. (25) one obtains a simplified equation which, by substituting $f(z) = u(z)^4$, can be further reduced to a simple Airy differential equation

$$zu(z) + 2ku''(z) = 0 \quad (27)$$

with the general solution

$$u(z) = \begin{cases} c_1 \text{Ai}\left(\frac{-z}{(2k)^{1/3}}\right) + c_2 \text{Bi}\left(\frac{-z}{(2k)^{1/3}}\right) & \text{for } k \geq 0 \\ c_1 \text{Ai}\left(\frac{z}{(-2k)^{1/3}}\right) + c_2 \text{Bi}\left(\frac{z}{(-2k)^{1/3}}\right) & \text{for } k < 0 \end{cases} \quad (28)$$

where $\text{Ai}(z)$ and $\text{Bi}(z)$ are Airy functions (see for instance [22]). For given $|k|$ the two solutions differ only by a reflection $z \rightarrow -z$ so that for the rest of this section we can restrict ourselves to the case $k > 0$ (which corresponds to a negative nonlinear term, i.e. $\lambda < 0$).

The two integration constants have to be chosen such that $f(z)$ is properly normalized and the appropriate boundary conditions are satisfied. For a free interface, the scaling function $f(z)$ has to vanish for $z \rightarrow \pm\infty$ in such a way that all of its moments are finite, hence $c_2 = 0$. Surprisingly, the remaining solution *oscillates* for $z \rightarrow \infty$ and does not vanish fast enough to yield finite moments. We conclude that the physically meaningful solution extends from $z = -\infty$ to the first root of the Airy function z_0 (where $f(z_0) = f'(z_0) = f''(z_0) = 0$) and vanishes elsewhere. The solution for the

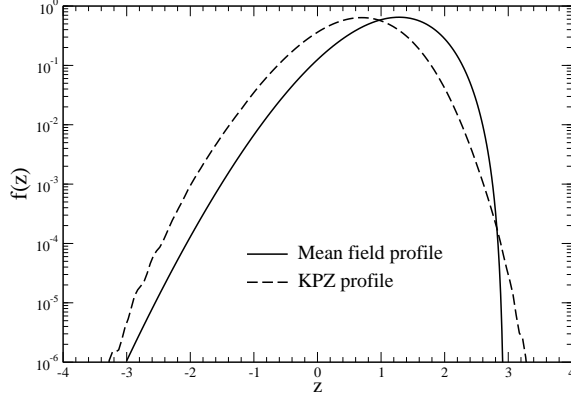


Figure 4: Rescaled skewed profile obtained within the mean field approximation (solid line) compared to the numerically determined KPZ profile using the single step model with $p = 1$. Differences between the two profiles can be highlighted by their mean value a_1 . The mean value controls the KPZ excess velocity and it is known to scale as $\lambda^{1/3}$. While the mean field profile is characterized by $a_1 \simeq 1.13184$, direct numerical estimate renders $a_{1,SSM} \simeq 0.60$. The vertical axis is plotted in a logarithmic scale.

free interface therefore reads

$$f(z) = \begin{cases} \frac{1}{\mathcal{N}} \left[\text{Ai}\left(\frac{-z}{(2k)^{1/3}}\right) \right]^4 & \text{for } -\infty < z < z_0 \\ 0 & \text{for } z_0 \leq z < +\infty \end{cases} \quad (29)$$

where $z_0 \simeq 2.94583 k^{1/3}$ and $\mathcal{N} \simeq 0.127153 k^{1/3}$. As it can be seen, the parameter k , describing the strength of the KPZ nonlinearity, appears here as a simple metric factor in the scaling function. Notice that by Eq. (26) the height unit Δ has been absorbed in k .

Figure 4 shows the solution (29) in comparison with the numerically determined profile of a freely roughening KPZ interface. Although the two curves are different due to the approximative character of the mean field theory, they share essential qualitative features. To quantify those it is instructive to compute the skewness

$$S = \frac{c_3}{c_2^{3/2}} = \frac{a_3 - 3a_2a_1 + 2a_1^3}{(a_2 - a_1^2)^{3/2}}, \quad (30)$$

where c_n is the n^{th} moment of the height probability distribution and $a_n = \int_{-\infty}^{z_0} dz z^n f(z)$ denotes the n^{th} central moment of the rescaled profile $f(z)$.

The skewness is expected to be universal in modulo for all KPZ growth processes, with its sign being equal to the sign of the nonlinear term. Known numerical estimates [23] give the value $|S_{\text{KPZ}}| = 0.28 \pm 0.04$, which is in good agreement with the theoretically computed skewness for the PNG model⁴ [8],

$$S_{\text{PNG}} \simeq 0.2935 . \quad (31)$$

Mean field theory, on the other hand, renders the value

$$S_{\text{MF}} \simeq \pm 0.465970 , \quad (32)$$

where the positive (negative) sign correspond to the case $k < 0$ ($k > 0$). Although this value is different from direct numerical estimates, it has the correct sign and the same order of magnitude, showing that the mean field theory captures qualitatively the influence of the nonlinear KPZ term.

Surprisingly, the mean field theory predicts that the interface profile is asymptotically bounded for $z > 0$ at a *finite* value z_0 . This means that within mean field the advancing front of the distribution exhibits a sharp cutoff rather than a stretched exponential tail. However, on the opposite side, where z is negative, the profile does indeed decay as a stretched exponential:

$$f(z) \sim |z|^{-1/4} e^{-\frac{2}{3\sqrt{2k}}|z|^{3/2}} \quad (z \rightarrow -\infty) \quad (33)$$

This result suggests that $\eta_+ = 3/2$, which coincides with the theoretical value predicted in the context of directed polymers and of the PNG model. On the other hand, η_- does not exist within the mean field approximation, which therefore fails to correctly describe large negative (w.r.t. the sign of the KPZ nonlinearity) height fluctuations.

4 Roughening in the vicinity of a wall

We now modify the single step model and the associated mean field equations in order to incorporate the presence of a hard core wall. Our aim is to determine the surface exponent θ introduced in Eq. (6) within the mean field approximation. In terms of the continuous height variable h , the density of pinned sites can be defined as the integral of the height probability distribution between the hard core wall and some arbitrary small height level h_0 ,

⁴The PNG model is characterized by a positive nonlinear term.

i.e.,

$$\rho(t) = \int_{vt}^{h_0+vt} P(h,t) dh = \int_0^{h_0 t^{-\gamma}} f(z) dz \quad , \quad (34)$$

so that the surface exponent is completely determined by the behavior of the scaling function $f(z)$ for $z \ll 1$.

In order to formulate the appropriate boundary condition in the continuum limit, one has to resort to the discrete formulation of the problem. Following the approach outlined in Ref. [24], the wall is initially located at zero height and moves *discontinuously* with the average velocity $v = p - \frac{1}{2}$, its actual height being given by $n_0(t) = \lfloor vt \rfloor$ (where $\lfloor \cdot \rfloor$ indicates the integer part). The interface is restricted to evolve above the wall, i.e.

$$P_n(t) = 0 \quad (n \leq n_0(t)) \quad (35)$$

so that the mean field equations (13) have to be modified accordingly close to the wall. In particular there is no probability current between level n_0 and $n_0 + 1$, so that

$$\frac{\partial}{\partial t} P_{n_0+1}(t) = -J_{n_0+1, n_0+2}(t) \quad , \quad (36)$$

while Eq. (15) still holds for $n > n_0 + 1$. Depending on p one has to distinguish three different cases. If $v > 0$ the wall advances by one unit in time intervals $\Delta t = 1/v$, flipping up all local minima at level $n_0 + 1$ by two units. This means that P_{n_0+1} is increased by P_{n_0} while P_{n_0} is set zero. If $v < 0$ the wall retracts by one unit in time intervals $\Delta t = |1/v|$, allowing height level n_0 , which was previously set to zero, to become nonzero during the subsequent evolution. Finally, for $v = 0$ the wall does not move, i.e. $n_0 = 0$ for all times t . The moving wall makes it difficult to specify the correct boundary conditions, so out of equilibrium we will derive them in the special cases $p = 1$ and $p = 0$, where the KPZ nonlinearity is maximal. Our reasoning, which once more relies on a series expansion in the proximities of the wall, shows that both a pushing ($p = 1$) and a retracting ($p = \frac{1}{2}$) wall impose a Dirichlet boundary condition for the scaling function. Surprisingly, it turns out that a retracting wall ($p = 0$) does not fix any boundary condition for the scaling function, which is free to assume any finite value at wall level, thus justifying the high density of pinned sites which is numerically observed in the case $\lambda > 0$ (see Fig. 2). General scaling arguments suggest that results obtained for $p = 1$ ($p = 0$) hold for any $p > 1/2$ ($p < 1/2$).

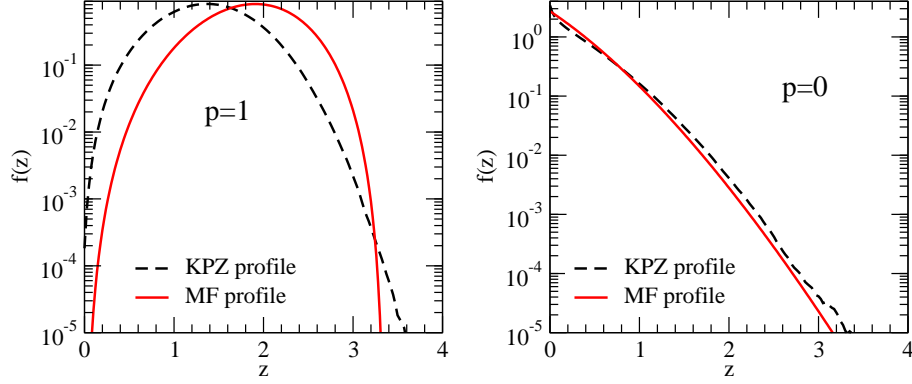


Figure 5: Rescaled mean field profiles in the presence of a hard core wall compared to the corresponding profiles in the SSM for $p = 1$ (left panel) and $p = 0$ (right panel). The vertical axis have been plotted in a logarithmic scale.

4.1 Depinning in the equilibrium case $p = 1/2$

In the equilibrium case we have $v = 0$ so that the wall does not move. As shown in the Appendix, a wall at zero height imposes a Dirichlet boundary condition $f(0) = 0$. Obviously, the only solution satisfying this boundary condition is Eq. (24)

$$f_{\text{eq}}^{\text{bound}}(z) = \frac{2^{5/4}}{\Delta^3 \sqrt{\pi}} z^2 \exp\left(-\frac{z^2}{\Delta^2 \sqrt{2}}\right) \quad (37)$$

which has been normalized here over the positive real axis. With this solution we can immediately read off the surface exponent from Eq. (34),

$$\theta_{\text{EW}}^{\text{MF}} = 3/4. \quad (38)$$

We note that this value coincides exactly with the known exponent for Edwards Wilkinson growth next to a wall.

4.2 Depinning in the non-equilibrium case $p = 1$

For $p > 1/2$ the wall advances discontinuously which makes it more difficult to specify the boundary conditions. As shown in the Appendix, in this case the co-moving wall again leads to a Dirichlet boundary condition $f(0) = 0$ for the scaling function. According to Eq. (28) the corresponding solution

then reads

$$f(z) = \begin{cases} \frac{1}{\mathcal{N}} \left[\text{Ai}\left(\frac{-z}{(2k)^{1/3}}\right) - \frac{1}{\sqrt{3}} \text{Bi}\left(\frac{-z}{(2k)^{1/3}}\right) \right]^4 & \text{for } 0 \leq z < z_1 \\ 0 & \text{for } z_1 \leq z < +\infty \end{cases} \quad (39)$$

where $z_1 \simeq 3.32426 k^{1/3}$ is the first positive root of the scaling function $f(z)$ (where also its first three derivatives vanish) and $\mathcal{N} \simeq 0.133454 k^{1/3}$. As in the free case the profile exhibits a sharp cutoff (see Fig. 5), although at a different value of z . Since $f(z) \sim z^4$ for $z \rightarrow 0$, the surface exponent is given by

$$\theta_{p=1}^{\text{MF}} = 4/3, \quad (40)$$

This value has to be compared with the numerical estimate $\theta_{\lambda < 0}^{\text{KPZ}} = 1.1(1)$ in Eq. (6).

4.3 Depinning in the non-equilibrium case $p = 0$

For $p < 1/2$ the wall moves discontinuously backward. As shown in the Appendix, this situation is special in so far as the retracting wall does not specify any boundary condition on the scaling function, allowing $f(0)$ to be nonzero. In fact, according to Eq. (28) the only normalizable solution is given by

$$f(z) = \frac{1}{\mathcal{N}} \left[\text{Ai}\left(\frac{z}{(2k)^{1/3}}\right) \right]^4 \quad (41)$$

with the normalization constant $\mathcal{N} \simeq 0.00584355 k^{1/3}$. Since $f(0) > 0$ the surface exponent is given by

$$\theta_{p=0}^{\text{MF}} = 1/3, \quad (42)$$

This value has to be compared with the numerical estimate $\theta_{\lambda > 0}^{\text{KPZ}} = 0.22(2)$ in Eq. (6).

5 Conclusions

In this paper we presented a mean field theory for 1+1 dimensional nonlinear growth processes evolving according to a KPZ equation. It is worth stressing that our approach does not neglect all types of fluctuations, as the term mean field usually suggests. Instead our equations retain information about the

one dimensional spatial structure and the height restriction $h_i - h_{i+1} = \pm 1$. Therefore, our mean field theory is not expected to hold exactly above some upper critical dimension, rather it serves as an approximation for the one dimensional case. Although we neglect spatial correlation between local slopes of the roughening interface, the theory has a very predictive power. Its success can be ascribed to a fluctuation dissipation theorem which is known to hold only for the 1+1 dimensional case [5]. Moreover, the mean field equations presented herein have been derived in the thermodynamic limit $L \rightarrow \infty$, and thus they are suited for probing finite time behavior, i.e. $t \ll L^z$.

Our approach is therefore successful in predicting a power law decay for the density of interfacial sites pinned to the substrate, thus supporting previous numerical studies of the non-equilibrium case. Although the mean field surface exponents θ differ from the numerically estimated values, our theory correctly reproduces the dramatic difference between the surface behavior in the presence of a negative or positive nonlinear KPZ term.

For what concerns the finite-time bulk properties of a KPZ interface, our simple mean field theory correctly reproduces the exact value for the roughening exponent $\gamma = 1/3$ and the skewed nature of the finite time height probability distribution. We found that one of the two tails of such a distribution decays as a stretched exponential with exponent $\eta_+ = 3/2$, thus confirming previous results obtained in the context of directed polymers and for the PNG model. While our theory successfully predicts large *positive* (w.r.t. to the sign of the KPZ nonlinearity) height fluctuations, it fails in describing large *negative* ones, exhibiting a sharp cut-off for the negative tail. It is therefore interesting to note that a simple scaling argument [9] proposed in the context of the PNG model directly relates η_+ to the roughening exponent, i.e. $\eta_+ = 1/(1 - \gamma)$, while no corresponding argument can be worked out for η_- . This is due to the fact that height fluctuations with the same sign w.r.t. the KPZ nonlinear term manifest as “bumps” (or “holes”) which grows laterally, while height fluctuations with the opposite sign manifest as “holes” (or “bumps”) which shrink laterally.

It is also worth noticing that the mean field theory reproduces correctly almost all bulk nonlinear critical properties at finite times, while it gives only approximate results for the surface exponent θ . This is an indication that the substrate introduces spatial correlations between local slopes at low height levels. It is our belief that a detailed study of the SSM with a hard substrate may eventually lead to the exact analytical knowledge of critical

depinning properties. It would also be interesting to find out whether the methods introduced in Ref. [8] can be applied to the problem of a KPZ interface with a wall.

Finally, equilibrium results are correctly reproduced as a marginal case, and small out of equilibrium corrections eventually leads to full KPZ behavior.

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A Boundary conditions in the presence of a hard core wall

In order to solve the mean field equation (18) when a hard core wall is imposed, it is necessary to go back to the discrete formulation of the problem and to consider the proper evolution equation for the height probability distribution (36) close to the wall. In the following we analyze the special cases $v = 0$ and $v = \pm \frac{1}{2}$ in order to derive the corresponding boundary conditions for the scaling function $f(z)$.

A.1 $p = \frac{1}{2}, v = 0$

In this case the wall does not move and $n_0 = 0$ for all times t . According to Eq. (36) the density of contact points $P_1(t)$ then evolves as

$$\frac{\partial}{\partial t} P_1(t) = -J_{1,2}(t), \quad (43)$$

where

$$J_{1,2}(t) = \frac{P_1(t)}{2} - \frac{P_2(t)^2}{2(P_2(t) + P_3(t))} = 0. \quad (44)$$

In the limit $t \rightarrow \infty$ this equation implies a Dirichlet boundary condition $f(0) = 0$. To see this let us assume that $f(0) \neq 0$ with $|f'(0)| < \infty$. Assuming EW scaling the first three probabilities would be given by $P_1(t) \simeq P_2(t) \simeq P_3(t) \simeq t^{-1/4} f(0)$ (where we expanded the scaling function f around $z = 0$ keeping only the leading term), giving rise to a current $J_{1,2}(t) \simeq \frac{1}{4} t^{-1/4} f(0)$. Since the l.h.s. scales as $t^{-5/4}$, Eq. (43) cannot hold unless $f(0) = 0$.

A.2 $p = 1, v = \frac{1}{2}$

For $p = 1$ one has $\Delta t = 2$, i.e. the wall advances by one unit after every second time step. To find the appropriate boundary condition for $f(z)$ (now assuming KPZ scaling with $\gamma = 1/3$), let us first consider the continuous temporal evolution between two advancements when the wall is fixed at some height $n_0 \lfloor -t_0/2 \rfloor$. As in the previous case the probabilities $P_n(t)$ with $n \leq n_0$ vanish. According to Eqs. (14), (15) and (36), the height probability distribution at the first two levels above the wall obeys to the differential equations

$$\begin{aligned} \frac{\partial}{\partial t} P_{n_0+1}(t) &= -P_{n_0+1}(t) \\ \frac{\partial}{\partial t} P_{n_0+2}(t) &= +P_{n_0+1}(t) - \frac{P_{n_0+2}^2}{P_{n_0+1} + P_{n_0+2}}. \end{aligned} \quad (45)$$

Let us again suppose that $f(0) \neq 0$ and $|f'(0)| < \infty$, i.e., just after the advancement of the wall at time t_0 we assume that to leading order $P_{n_0+1}(t_0) \simeq P_{n_0+2}(t_0) \simeq t_0^{-1/3} f(0) =: c(t_0)$. Iterating the differential equations (45) over two time steps, and by assuming $c(t) \simeq c(t_0)$ for $t_0 \leq t \leq t_0 + 2$, one obtains $P_{n_0+1}(t_0 + 2) \simeq c(t_0)e^{-2}$. On the other hand, by numerically solving the differential equation for level $n_0 + 2$

$$P_{n_0+2}(t) \simeq +c(t_0)e^{-(t-t_0)} - \frac{P_{n_0+2}^2}{c(t_0)e^{-(t-t_0)} + P_{n_0+2}} \quad (46)$$

one gets $P_{n_0+2}(t_0 + 2) \approx 0.596 c(t_0)$. At time $t_0 + 2$ the wall advances by one unit, i.e. all local minima at height n_0 are flipped upwards, meaning that $P_{n_0+1}(t_0 + 2)$ is first added to $P_{n_0+2}(t_0 + 2)$ and then set to zero. Just after advancement $P_{n_0+2}(t_0 + 2) \approx 0.732 c(t_0)$ which is in contradiction the assumption unless $f(0) = 0$. Hence for $p = 1$ the wall imposes again a Dirichlet boundary condition.

A.3 $p = 0, v = -\frac{1}{2}$

For $p < 0$ the wall retracts by one unit after every second time step. Let us first consider the continuous temporal evolution between two moves when the wall is fixed at height $n_0 = \lfloor -t_0/2 \rfloor$. As usual the probabilities $P_n(t)$ with $n \leq n_0$ vanish and the the first two levels, where the height probability

distribution is nonzero, evolve according to the differential equations

$$\begin{aligned}\frac{\partial}{\partial t}P_{n_0+1}(t) &= \frac{P_{n_0+2}^2}{P_{n_0+2} + P_{n_0+3}} \\ \frac{\partial}{\partial t}P_{n_0+2}(t) &= -\frac{P_{n_0+2}^2}{P_{n_0+2} + P_{n_0+3}} + \frac{P_{n_0+3}^2}{P_{n_0+3} + P_{n_0+4}}\end{aligned}\tag{47}$$

Just after retraction level $n_0 + 1$ can be visited by the interface for the first time, hence $P_{n_0+1}(t_0)$ is initially zero and becomes nonzero as time evolves. Assuming that $f(0) \neq 0$ and $|f'(0)| < \infty$, to leading order the other probabilities have the initial values $P_{n_0+2}(t_0) \simeq P_{n_0+3}(t_0) \simeq t_0^{-1/3}f(0) =: c(t_0)$. Iterating the differential equations (45) over two time steps one obtains $P_{n_0+1}(t_0 + 2) \simeq c(t_0)$ while the higher levels remain unchanged. Thus the almost constant probability distribution in the vicinity the wall is simply 'extended' to the new level that becomes available by retraction of the wall, meaning that a nonzero value $f(0) \neq 0$ and $|f'(0)| < \infty$ is consistent with the equations (47). Loosely speaking the wall retracts so quickly that it does not impose a specific boundary condition, allowing $f(0)$ to take any positive value. The equations are built in such a way that this value is simply copied from the following height level.

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